

Global Attitude Estimation using Uncertainty Ellipsoids

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Abstract—Attitude estimation is often a prerequisite for control of the attitude or orientation of mechanical systems. Current attitude estimation algorithms use coordinate representations for the group of rigid body orientations. All coordinate representations of the group of orientations have associated problems. While minimal coordinate representations exhibit kinematic singularities for large rotations, non-minimal coordinates like quaternions require satisfaction of extra constraints. A deterministic attitude estimation problem for a rigid body with bounded measurement errors is considered here. An attitude estimation algorithm that globally minimizes the attitude estimation error, is obtained. Assuming that the initial attitude, the initial angular velocity and measurement noise lie within given ellipsoidal bounds, an uncertainty ellipsoid that bounds the attitude and the angular velocity of the rigid body is obtained. The center of the uncertainty ellipsoid provides point estimates, and the size of the uncertainty ellipsoid measures the accuracy of the estimates. The point estimates, and the uncertainty ellipsoids are propagated using a Lie group variational integrator, and its linearization, respectively. The attitude estimation is optimal in the sense that the attitude estimation error and the size of the uncertainty ellipsoid is minimized.

I. INTRODUCTION

Attitude estimation is often a prerequisite for controlling aerospace and underwater vehicles, mobile robots, and other mechanical systems moving in space. Hence, attitude estimation may be used in spacecraft and aircraft, unmanned vehicles and robots, including walking robots. In this paper, we look at the attitude estimation problem for the uncontrolled dynamics of a rigid body in an attitude-dependent force potential. The estimation scheme we present has the following important features: (1) the attitude is globally represented without using any coordinate system, (2) the filter obtained is not a Kalman or extended Kalman filter, and (3) the attitude and angular velocity measurement errors are assumed to be bounded, with ellipsoidal uncertainty bounds. The static attitude estimation using a global attitude representation is based on [1]. Such a global representation has been recently used for partial attitude estimation with a linear dynamics model in [2].

The attitude determination problem for a rigid body from vector measurements was first posed in [3]. A sample of the literature in spacecraft attitude estimation can be found in [4], [5], [6], [7], [8]. Applications of attitude estimation to unmanned vehicles and robots can be found in [2], [9], [10], [11]. Most existing attitude estimation schemes use

coordinate representations of the attitude. As is well known, minimal coordinate representations of the rotation group, like Euler angles, Rodrigues parameters, and modified Rodrigues parameters (see [12]), usually lead to geometric or kinematic singularities. Non-minimal coordinate representations, like the quaternions used in the quaternion estimation (QUEST) algorithm and its several variants ([4], [8], [13]), have their own associated problems. Besides the extra constraint of unit norm that one needs to impose on the quaternion, the quaternion representation for a given rotation depends on the sense of rotation used to define the principal angle, and hence can be defined in one of two ways.

A brief outline of this paper is given here. In Section II, the attitude determination problem for vector measurements with measurement noise is introduced, and a global attitude determination algorithm which minimizes the attitude estimation error is presented. In Section III, the attitude dynamics and dynamic estimation problem is formulated, and an algorithm to numerically integrate the dynamics is presented. Section IV presents the attitude estimation scheme with attitude and angular velocity measurements. Section V presents some simulation results followed by conclusions in Section VI.

II. ATTITUDE DETERMINATION

A. Attitude determination from vector observations

Attitude of a rigid body is defined by the orientation of a body fixed frame with respect to a reference frame, and the attitude is represented by a rotation matrix that is a 3×3 orthogonal matrix with determinant of 1. Rotation matrices have a group structure denoted by $SO(3)$. The group operation of $SO(3)$ is matrix multiplication, and its action on \mathbb{R}^3 takes a vector represented in body fixed frame into the reference frame by matrix multiplication.

We denote the known direction vector of the i th point in the reference frame as $e^i \in \mathbb{S}^2$, and the corresponding vector represented in the body fixed frame as $b^i \in \mathbb{S}^2$. These direction vectors are normalized so that they have unit lengths. The e^i and b^i are related by a rotation matrix $C \in SO(3)$ that defines the rigid body attitude;

$$e^i = Cb^i,$$

for all $i \in \{1, 2, \dots, m\}$, where m is the number of measurements. We assume that e^i is known accurately and b^i is measured by sensors in the body fixed frame. Let the

measured direction vectors (with sensor errors) be denoted $\tilde{b}^i \in \mathbb{S}^2$, and let an estimate of the rotation matrix be denoted $\hat{C} \in \text{SO}(3)$. The estimation error is given by

$$e^i - \hat{C}\tilde{b}^i.$$

The attitude determination problem consists of finding an estimate $\hat{C} \in \text{SO}(3)$, and is given by the following weighted least squares problem:

$$\begin{aligned} \min_{\hat{C}} \mathcal{J} &= \frac{1}{2} \sum_{i=1}^m w_i (e^i - \hat{C}\tilde{b}^i)^T (e^i - \hat{C}\tilde{b}^i), \\ &= \frac{1}{2} \text{tr} \left[(E - \hat{C}\tilde{B})^T W (E - \hat{C}\tilde{B}) \right], \quad (1) \\ \text{subject to } \hat{C} &\in \text{SO}(3), \end{aligned}$$

where $E = [e^1, e^2, \dots, e^m] \in \mathbb{R}^{3 \times m}$, $\tilde{B} = [\tilde{b}^1, \tilde{b}^2, \dots, \tilde{b}^m] \in \mathbb{R}^{3 \times m}$, and $W = \text{diag}[w^1, w^2, \dots, w^m] \in \mathbb{R}^{m \times m}$ has weight factors for each measured vector.

This problem is known as Wahba's problem [3]. The solution in terms of quaternions, known as the QUEST algorithm, is presented in [7]. A solution without using generalized attitude coordinates is given in [1]. A necessary condition for optimality of (1) is given by

$$L^T \hat{C} = \hat{C}^T L, \quad (2)$$

where $L = EW\tilde{B}^T \in \mathbb{R}^{3 \times 3}$.

The following result, which is proved in [1], gives an unique estimate $\hat{C} \in \text{SO}(3)$ of the attitude matrix that solves the attitude determination problem (1).

Theorem 1: The unique minimizing solution to the attitude determination problem (1) is given by

$$\hat{C} = SL, \quad S = Q\sqrt{(RR^T)^{-1}}Q^T, \quad (3)$$

where

$$L = QR, \quad Q \in \text{SO}(3), \quad (4)$$

and R is upper triangular and invertible; this is the QR decomposition of L . The symmetric positive definite (principal) square root is used in (3).

The proof is based on the fact that \mathcal{J} is a Morse function, i.e., its critical points are non-degenerate. From the Morse lemma [14], we conclude that these non-degenerate critical points are isolated, and hence the estimate given by (3) uniquely minimizes the attitude estimation error.

B. Estimation with bounded state uncertainties

A stochastic state estimator requires probabilistic models for the state uncertainty and the noise, which are often not available. Assumptions are usually made on the statistics of disturbance and noise processes, in order to make the estimation problem mathematically tractable. In many practical situations such idealized assumptions are not appropriate, and may cause poor estimation performance [15]. An alternative deterministic approach is to specify bounds on the uncertainty and the measurement noise without any

assumptions on their distribution. Noise bounds are available in many cases, and such a deterministic estimation scheme is robust to the noise distribution. An efficient but flexible way to describe the bounds is using ellipsoidal sets, referred to as uncertainty ellipsoids.

This deterministic estimation procedure for a 2 dimensional system is illustrated in Fig. 1, where the left figure shows time evolution of an uncertainty ellipsoid, and the right figure shows a cross section at a fixed time when the state is measured. At the k th time step, the state is bounded by an uncertainty ellipsoid centered at \hat{x}_k . This initial ellipsoid evolves over time. Depending on the dynamics of the system, the size and the shape of the tube are changed. At the $k+1$ th time step, the predicted uncertainty ellipsoid is centered at \hat{x}_{k+1}^f . The state is then measured by sensors, and another ellipsoidal bound on the state is obtained by the measurements. The measured uncertainty ellipsoid is centered at \hat{x}_{k+1}^m . The state lies in the intersection of the two ellipsoids. In the estimation procedure, we find a new ellipsoid that contains the intersection, which is shown in the right figure. The center of the new ellipsoid, \hat{x}_{k+1} is considered as a point estimate at time step $k+1$, and the magnitude of the new uncertainty ellipsoid measures the accuracy of the estimation. This deterministic estimation is optimal in the sense that the size of the new ellipsoid is minimized.

A deterministic estimation process based on set theoretic results was developed in [16]. Optimal deterministic estimation is considered in [17] and [18], where an analytic solution for the minimum ellipsoid that contains a union or an intersection of ellipsoids is obtained.

III. ATTITUDE DYNAMICS AND DYNAMIC ATTITUDE ESTIMATION

A. Equations of motion

We now consider dynamic state estimation of the attitude dynamics of a rigid body in a potential $U(C) : \text{SO}(3) \mapsto \mathbb{R}$ determined by the attitude, $C \in \text{SO}(3)$. A spacecraft on a circular orbit including gravity gradient effects [19], or a 3D pendulum [20] can be so modeled. The continuous equations of motion are given by

$$J\dot{\omega} + \omega \times J\omega = M, \quad (5)$$

$$\dot{C} = CS(\omega), \quad (6)$$

where $J \in \mathbb{R}^{3 \times 3}$ is the moment of inertia matrix of the rigid body, $\omega \in \mathbb{R}^3$ is the angular velocity of the body expressed in the body fixed frame, and $S(\cdot) : \mathbb{R}^3 \mapsto \mathfrak{so}(3)$ is a skew mapping defined such that $S(x)y = x \times y$ for all $x, y \in \mathbb{R}^3$. $M \in \mathbb{R}^3$ is the moment due to the potential. The moment is determined by $S(M) = \frac{\partial U}{\partial C}^T R - C^T \frac{\partial U}{\partial C}$, or more explicitly,

$$M = r_1 \times v_{r_1} + r_2 \times v_{r_2} + r_3 \times v_{r_3}, \quad (7)$$

where $r_i, v_{r_i} \in \mathbb{R}^{1 \times 3}$ are the i th row vectors of C and $\frac{\partial U}{\partial C}$, respectively. The derivation of the above equations can be found in [20].

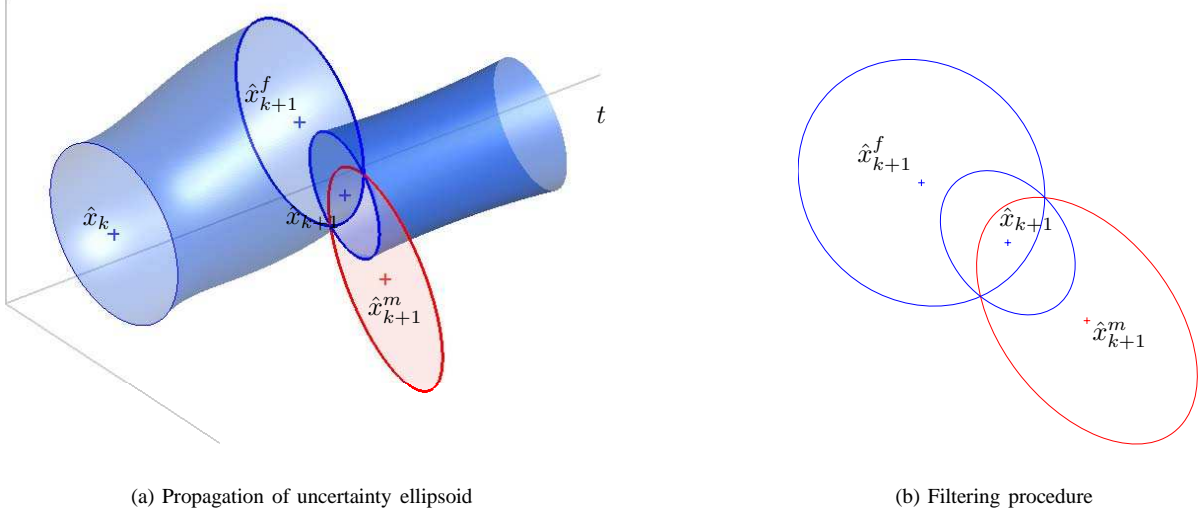


Fig. 1. Uncertainty ellipsoids

General numerical integration methods, including the popular Runge-Kutta schemes, typically preserve neither first integrals nor the characteristics of the configuration space, $\text{SO}(3)$. In particular, the orthogonal structure of the rotation matrices is not preserved numerically. To resolve these problems, a Lie group variational integrator for the attitude dynamics of a rigid body is proposed in [20]. This Lie group variational integrator is described by the discrete time equations.

$$hS(J\omega_k + \frac{h}{2}M_k) = F_k J_d - J_d F_k^T, \quad (8)$$

$$C_{k+1} = C_k F_k, \quad (9)$$

$$J\omega_{k+1} = F_k^T J\omega_k + \frac{h}{2}F_k^T M_k + \frac{h}{2}M_{k+1}, \quad (10)$$

where $J_d \in \mathbb{R}^3$ is a nonstandard moment of inertia matrix defined by $J_d = \frac{1}{2}\text{tr}[J]I_{3 \times 3} - J$, and $F_k \in \text{SO}(3)$ is the relative attitude over an integration step. The constant $h \in \mathbb{R}$ is the integration step size, and the subscript k denotes the k th integration step. This integrator yields a map $(C_k, \omega_k) \mapsto (C_{k+1}, \omega_{k+1})$ by solving (8) to obtain $F_k \in \text{SO}(3)$ and substituting it into (9) and (10) to obtain C_{k+1} and ω_{k+1} .

Since this integrator does not use a local parameterization, the attitude is defined globally without singularities. It preserves the orthogonal structure of $\text{SO}(3)$ because the rotation matrix is updated by a multiplication of two rotation matrices in (9). This integrator is obtained from a discrete variational principle, and it exhibits the characteristic symplectic and momentum preservation properties, and good energy behavior characteristic of variational integrators. We use (8), (9), and (10) in the following development of the attitude estimator.

B. Uncertainty Ellipsoid

The configuration space of the attitude dynamics is $\text{SO}(3)$, so the state evolves in $\text{TSO}(3)$. Thus the corresponding

uncertainty ellipsoid is a submanifold of $\text{TSO}(3)$. An uncertainty ellipsoid centered at $(\hat{C}, \hat{\omega})$ is induced from an uncertainty ellipsoid in \mathbb{R}^6 , using the Lie algebra $\mathfrak{so}(3)$;

$$\begin{aligned} \mathcal{E}(\hat{C}, \hat{\omega}, P) &= \left\{ C \in \text{SO}(3), \omega \in \mathbb{R}^3 \mid \begin{bmatrix} \zeta^T & \delta\omega^T \end{bmatrix} P^{-1} \begin{bmatrix} \zeta \\ \delta\omega \end{bmatrix} \leq 1 \right\}, \\ &= \left\{ C \in \text{SO}(3), \omega \in \mathbb{R}^3 \mid \begin{bmatrix} \zeta \\ \delta\omega \end{bmatrix} \in \mathcal{E}_{\mathbb{R}^6}(0_6, P) \right\}, \end{aligned} \quad (11)$$

where $S(\zeta) = \logm(\hat{C}^T C) \in \mathfrak{so}(3)$, $\delta\omega = \omega - \hat{\omega} \in \mathbb{R}^3$, and $P \in \mathbb{R}^{6 \times 6}$ is a symmetric positive definite matrix. Equivalently, an element $(C, \omega) \in \mathcal{E}(\hat{C}, \hat{\omega}, P)$ can be written as

$$\begin{aligned} C &= \hat{C} e^{S(\zeta)}, \\ \omega &= \hat{\omega} + \delta\omega, \end{aligned}$$

for $x = \begin{bmatrix} \zeta^T & \delta\omega^T \end{bmatrix}^T \in \mathbb{R}^6$ satisfying $x^T P^{-1} x \leq 1$.

C. Measurement error model

We give the measurement error models for the direction vector and for the angular velocity. The direction vector $b^i \in \mathbb{S}^2$ is in the body fixed frame, and let $\tilde{b}^i \in \mathbb{S}^2$ denote the corresponding measured directions. Since we only measure directions, it is inappropriate to express the measurement error by a vector difference. Instead, we model it by rotation of the measured direction;

$$\begin{aligned} b^i &= e^{S(\nu^i)} \tilde{b}^i, \\ &\simeq \tilde{b}^i + S(\nu^i) \tilde{b}^i, \end{aligned} \quad (12)$$

where $\nu^i \in \mathbb{R}^3$ is the sensor error, which represents the Euler axis of rotation vector from \tilde{b}^i to b^i , and $\|\nu^i\|$ is the corresponding rotation angle in radians. The second equality

assumes small measurement errors. The angular velocity measurement errors are modeled as

$$\omega_k = \tilde{\omega}_k + v_k, \quad (13)$$

where $\tilde{\omega}_k \in \mathbb{R}^3$ is the measured angular velocity, and $v_k \in \mathbb{R}^3$ is an additive error.

We assume that the initial conditions and the sensor noise are bounded by prescribed uncertainty ellipsoids.

$$(C_0, \omega_0) \in \mathcal{E}(\hat{C}_0, \hat{\omega}_0, P_0), \quad (14)$$

$$\nu_k^i \in \mathcal{E}_{\mathbb{R}^3}(0, S_k^i), \quad (15)$$

$$v_k \in \mathcal{E}_{\mathbb{R}^3}(0, T_k), \quad (16)$$

where $P_0 \in \mathbb{R}^{6 \times 6}$, $S_k^i, T_k \in \mathbb{R}^{3 \times 3}$ are symmetric positive definite matrices that define the shape and the size of the uncertainty ellipsoids.

IV. ATTITUDE ESTIMATION WITH ANGULAR VELOCITY SENSOR

In this section, we develop a deterministic estimator for the attitude and the angular velocity of a rigid body assuming that both attitude measurement and angular velocity measurements are available. The estimator consists of three stages; flow propagation, measurement, and filtered update. The propagation is to predict the uncertainty ellipsoid in the future. The measurement is to find an uncertainty ellipsoid in the state space using the measurements and the measurement error model. The filtered update finds a new estimate using the predicted uncertainty ellipsoid and the measured uncertainty ellipsoid.

The subscript k denotes the k -th discrete index. This may not coincide with measurement instants as we may resolve the evolution of the trajectory more frequently than the frequency of the measurements. This enables us to deal with measurements that are rather infrequent, with nontrivial attitude evolution between the measurements. The superscript f denotes the variables related to the flow update, and the superscript m denotes the variables related to the measurement update. $\tilde{\cdot}$ denotes a variable measured by sensors, and $\hat{\cdot}$ denotes an estimated variable.

A. Flow propagation

Suppose that the attitude and the angular momentum at the k th step lie in a given uncertainty ellipsoid:

$$(C_k, \omega_k) \in \mathcal{E}(\hat{C}_k, \hat{\omega}_k, P_k),$$

and suppose that new measurements are taken at the $k + l$ th time step.

The flow update obtains the the uncertainty ellipsoid at the $k + l$ th step using the given uncertainty ellipsoid at the k th step. We assume that the given uncertainty ellipsoid at the k th step is sufficiently small that the states in the uncertainty ellipsoid can be approximated by linearized equations of motion. This guarantees that the boundary of the state uncertainties at the $k + l$ th step remains an ellipsoid.

Center: For the given center, $(\hat{C}_k, \hat{\omega}_k)$, the center of the uncertainty ellipsoid $(\hat{C}_{k+l}^f, \hat{\omega}_{k+l}^f)$ is obtained from the discrete equations of motion, (8), (9), and (10):

$$hS(J\hat{\omega}_k + \frac{h}{2}\hat{M}_k) = \hat{F}_k J_d - J_d \hat{F}_k^T, \quad (17)$$

$$\hat{C}_{k+l}^f = \hat{C}_k \hat{F}_k, \quad (18)$$

$$J\hat{\omega}_{k+l}^f = \hat{F}_k^T \hat{\omega}_k + \frac{h}{2} \hat{F}_k^T \hat{M}_k + \frac{h}{2} \hat{M}_{k+l}. \quad (19)$$

This integrator yields a map $(\hat{C}_k, \hat{\omega}_k) \mapsto (\hat{C}_{k+l}^f, \hat{\omega}_{k+l}^f)$, and this process can be repeated to find the center at the $k + l$ th step, $(\hat{C}_{k+l}^f, \hat{\omega}_{k+l}^f)$.

Uncertainty matrix: The uncertainty matrix is obtained by linearizing the above discrete equations of motion. At the $(k + l)$ th step, the state is given by perturbations from the center $(\hat{C}_{k+l}^f, \hat{\omega}_{k+l}^f)$ as

$$C_{k+l} = \hat{C}_{k+l}^f e^{S(\zeta_{k+l}^f)},$$

$$\omega_{k+l} = \hat{\omega}_{k+l}^f + \delta\omega_{k+l}^f,$$

for some $\zeta_{k+l}^f, \delta\omega_{k+l}^f \in \mathbb{R}^3$. Assume that the uncertainty ellipsoid at the k th step is sufficiently small. Then, $\zeta_{k+l}^f, \delta\omega_{k+l}^f$ are given by the following linear equations in [19]:

$$\begin{bmatrix} \zeta_{k+l}^f \\ \delta\omega_{k+l}^f \end{bmatrix} = \begin{bmatrix} \mathcal{A}_k^f & \mathcal{B}_k^f \\ \mathcal{C}_k^f & \mathcal{D}_k^f \end{bmatrix} \begin{bmatrix} \zeta_k \\ \delta\omega_k \end{bmatrix},$$

where $\mathcal{A}_k^f, \mathcal{B}_k^f, \mathcal{C}_k^f, \mathcal{D}_k^f \in \mathbb{R}^{3 \times 3}$ can be suitably defined. Equivalently, we rewrite the above equation as

$$x_{k+l}^f = A_k^f x_k,$$

where $x_k = [\zeta_k^T, \delta\omega_k^T]^T \in \mathbb{R}^6$, $A_k^f \in \mathbb{R}^{6 \times 6}$. Since $(C_k, \omega_k) \in \mathcal{E}(\hat{C}_k, \hat{\omega}_k, P_k)$, $x_k \in \mathcal{E}_{\mathbb{R}^6}(0, P_k)$ by the definition of the uncertainty ellipsoid given in (11), we can show that

$$A_k^f x_k \in \mathcal{E}_{\mathbb{R}^6}\left(0, A_k^f P_k (A_k^f)^T\right).$$

Thus, the uncertainty matrix at the $k + l$ th step is given by

$$P_{k+l}^f = A_k^f P_k (A_k^f)^T. \quad (20)$$

The above equation can be applied repeatedly to find the uncertainty matrix at the $k + l$ th step. In summary, the uncertainty ellipsoid at the $(k + l)$ th step is computed using (17), (18), (19), and (20) as:

$$(C_{k+l}, \omega_{k+l}) \in \mathcal{E}(\hat{C}_{k+l}^f, \hat{\omega}_{k+l}^f, P_{k+l}^f), \quad (21)$$

B. Measurement update

The measured attitude and angular velocity have uncertainties due to sensor errors. However, we can find a uncertainty bound on the states because we assume that the sensor errors are bounded by known uncertainty ellipsoids. The measurement update obtains an uncertainty ellipsoid in the state space using the measurements and the sensor error models.

Center: The center of the uncertainty ellipsoid, $(\hat{C}_{k+l}^m, \hat{\omega}_{k+l}^m)$ is obtained from the measurements. The

attitude is determined by measuring the directions to the known points in the inertial frame. Let the measured directions to the known points be $\tilde{B}_{k+l} = [\tilde{b}^1, \tilde{b}^2, \dots, \tilde{b}^m] \in \mathbb{R}^{3 \times m}$. Then, the attitude \hat{C}_{k+l}^m satisfies the following necessary and sufficient condition given in (2).

$$\left(\hat{C}_{k+l}^m\right)^T \tilde{L}_{k+l} - \tilde{L}_{k+l}^T \hat{C}_{k+l}^m = 0, \quad (22)$$

where $\tilde{L}_{k+l} = E_{k+l} W_{k+l} \tilde{B}_{k+l}^T \in \mathbb{R}^{3 \times 3}$. The solution of (22) is obtained by a QR factorization of \tilde{L}_{k+l} as given in Theorem 1.

$$\hat{C}_{k+l}^m = \left(Q \sqrt{(R R^T)^{-1}} Q^T\right) \tilde{L}_{k+l}, \quad (23)$$

where $Q \in \text{SO}(3)$ and $R \in \mathbb{R}^{3 \times 3}$ is upper triangular such that $\tilde{L}_{k+l} = QR$. The angular velocity is measured directly by sensors;

$$\hat{\omega}_{k+l}^m = \tilde{\omega}_{k+l}. \quad (24)$$

Uncertainty matrix: We can represent the actual state at the $k+l$ th step as follows:

$$C_{k+l} = \hat{C}_{k+l}^m e^{S(\zeta_{k+l}^m)}, \quad (25)$$

$$\omega_{k+l} = \hat{\omega}_{k+l}^m + \delta\omega_{k+l}^m, \quad (26)$$

for $\zeta_{k+l}^m, \delta\omega_{k+l}^m \in \mathbb{R}^3$. The uncertainty matrix is obtained by finding an ellipsoidal bound for $\zeta_{k+l}^m, \delta\omega_{k+l}^m$.

For the attitude, we transform the uncertainties in the directional sensors into the uncertainties in the rotation matrix by (22). The actual matrix of body direction vectors B_{k+l} and the actual attitude C_{k+l} also satisfy (23);

$$C_{k+l}^T L_{k+l} - L_{k+l}^T C_{k+l} = 0, \quad (27)$$

where $L_{k+l} = E_{k+l} W_{k+l} B_{k+l}^T \in \mathbb{R}^{3 \times 3}$. Using the identity, $S(x)A + A^T S(x) = S(\{\text{tr}[A] I_{3 \times 3} - A\} x)$ for $A \in \mathbb{R}^{3 \times 3}$, $x \in \mathbb{R}^3$, (27) can be written in the vector form

$$\begin{aligned} & \left\{ \text{tr} \left[\left(\hat{C}_{k+l}^m \right)^T \tilde{L}_{k+l} \right] - \left(\hat{C}_{k+l}^m \right)^T \tilde{L}_{k+l} \right\} \zeta_{k+l}^m \\ &= - \sum_{i=1}^m w_i \left\{ \text{tr} \left[\tilde{b}^i (e^i)^T \hat{C}_{k+l}^m \right] I_{3 \times 3} - \tilde{b}^i (e^i)^T \hat{C}_{k+l}^m \right\} \nu^i. \end{aligned}$$

Then, we obtain

$$\zeta_{k+l}^m = \sum_{i=1}^m \mathcal{A}_{k+l}^{m,i} \nu^i, \quad (28)$$

where

$$\begin{aligned} \mathcal{A}_{k+l}^{m,i} &= - \left\{ \text{tr} \left[\left(\hat{C}_{k+l}^m \right)^T \tilde{L}_{k+l} \right] - \left(\hat{C}_{k+l}^m \right)^T \tilde{L}_{k+l} \right\}^{-1} \\ & w_i \left\{ \text{tr} \left[\tilde{b}^i (e^i)^T \hat{C}_{k+l}^m \right] I_{3 \times 3} - \tilde{b}^i (e^i)^T \hat{C}_{k+l}^m \right\}. \end{aligned} \quad (29)$$

This equation expresses the error in the measured attitude as a linear combination of the directional sensor errors.

The perturbation of the angular velocity $\delta\omega_{k+l}^m$ is equal to the angular velocity measurement error v_{k+l} . Substituting (26) into (13), we obtain

$$\delta\omega_{k+l}^m = v_{k+l}. \quad (30)$$

Define the error states $x_{k+l}^m = \left[\left(\zeta_{k+l}^m \right)^T, \left(\delta\omega_{k+l}^m \right)^T \right]^T \in \mathbb{R}^6$. Using (28) and (30),

$$x_{k+l}^m = H_1 \sum_{i=1}^m \mathcal{A}_{k+l}^{m,i} \nu_{k+l}^i + H_2 v_{k+l},$$

where $H_1 = [I_{3 \times 3}, 0_{3 \times 3}]^T, H_2 = [0_{3 \times 3}, I_{3 \times 3}]^T \in \mathbb{R}^{6 \times 3}$ which expresses x_{k+l}^m as a linear combination of the sensor errors ν^i and v . From (15) and (16), each term on the right hand side is in the following uncertainty ellipsoids:

$$\begin{aligned} H_1 \mathcal{A}_{k+l}^{m,i} \nu_{k+l}^i &\in \mathcal{E}_{\mathbb{R}^6} \left(0, H_1 \mathcal{A}_{k+l}^{m,i} S_{k+l}^i \left(\mathcal{A}_{k+l}^{m,i} \right)^T H_1^T \right), \\ H_2 v_{k+l} &\in \mathcal{E}_{\mathbb{R}^6} \left(0, H_2 T_{k+l} H_2^T \right). \end{aligned}$$

The measurement update finds a minimal ellipsoid containing the vector sum of these uncertainty ellipsoids. Expressions for a minimal ellipsoid containing multiple ellipsoids are given in [17] and [18], and P_{k+l}^m is given by

$$\begin{aligned} P_{k+l}^m &= \left\{ \sum_{i=1}^m \sqrt{\text{tr} \left[H_1 \mathcal{A}_{k+l}^{m,i} S_{k+l}^i \left(\mathcal{A}_{k+l}^{m,i} \right)^T H_1^T \right]} \right. \\ &+ \left. \sqrt{\text{tr} \left[H_2 T_{k+l} H_2^T \right]} \right\} \left\{ \sum_{i=1}^m \frac{H_1 \mathcal{A}_{k+l}^{m,i} S_{k+l}^i \left(\mathcal{A}_{k+l}^{m,i} \right)^T H_1^T}{\sqrt{\text{tr} \left[H_1 \mathcal{A}_{k+l}^{m,i} S_{k+l}^i \left(\mathcal{A}_{k+l}^{m,i} \right)^T H_1^T \right]}} \right. \\ &+ \left. \frac{H_2 T_{k+l} H_2^T}{\sqrt{\text{tr} \left[H_2 T_{k+l} H_2^T \right]}} \right\}. \end{aligned} \quad (31)$$

In summary, the measured uncertainty ellipsoid at the $k+l$ th step is defined by (23), (24), and (31);

$$(C_{k+l}, \omega_{k+l}) \in \mathcal{E}(\hat{C}_{k+l}^m, \hat{\omega}_{k+l}^m, P_{k+l}^m). \quad (32)$$

C. Filtering procedure

The filtering procedure is to find a new uncertainty ellipsoid compatible with the predicted and the measured uncertainty ellipsoids. From (21) and (32), the state at $k+l$ th step lies in the intersection

$$(C_{k+l}, \omega_{k+l}) \in \mathcal{E}(\hat{C}_{k+l}^f, \hat{\omega}_{k+l}^f, P_{k+l}^f) \cap \mathcal{E}(\hat{C}_{k+l}^m, \hat{\omega}_{k+l}^m, P_{k+l}^m). \quad (33)$$

Since it is inefficient to describe an irregular subset like the intersection of two ellipsoids in the state space numerically, we find a minimal uncertainty ellipsoid containing the intersection. We omit the subscript $(k+l)$ in this subsection for convenience.

The measurement uncertainty ellipsoid, $\mathcal{E}(\hat{C}^m, \hat{\omega}^m, P^m)$, is identified by its center $(\hat{C}^m, \hat{\omega}^m)$, and the uncertainty ellipsoid in \mathbb{R}^6 :

$$(\zeta^m, \delta\omega^m) \in \mathcal{E}_{\mathbb{R}^6}(0_{6 \times 1}, P^m), \quad (34)$$

where $S(\zeta^m) = \log m(\hat{C}^m, {}^T C) \in \mathfrak{so}(3)$, $\delta\omega^m = \omega - \hat{\omega}^m \in \mathbb{R}^3$. Similarly, the predicted uncertainty ellipsoid, $\mathcal{E}(\hat{C}^f, \hat{\omega}^f, P^f)$, is identified by its center $(\hat{C}^f, \hat{\omega}^f)$, and the uncertainty ellipsoid in \mathbb{R}^6 :

$$(\zeta^f, \delta\omega^f) \in \mathcal{E}_{\mathbb{R}^6}(0_{6 \times 1}, P^f), \quad (35)$$

where $S(\zeta^f) = \log m(\hat{C}^f, {}^T C) \in \mathfrak{so}(3)$, $\delta\omega^f = \omega - \hat{\omega}^f \in \mathbb{R}^3$.

Define $\hat{\zeta}^{mf}, \delta\hat{\omega}^{mf} \in \mathbb{R}^3$ such that

$$\hat{C}^f = \hat{C}^m e^{S(\hat{\zeta}^{mf})}, \quad (36)$$

$$\hat{\omega}^f = \hat{\omega}^m + \delta\hat{\omega}^{mf}. \quad (37)$$

Thus, $\hat{\zeta}^{mf}, \delta\hat{\omega}^{mf}$ gives the difference between the centers of the two ellipsoids. Using (36) and (37) we get

$$\begin{aligned} C^f &= \hat{C}^m e^{S(\hat{\zeta}^{mf})} e^{S(\zeta^f)}, \\ &\simeq \hat{C}^m e^{S(\hat{\zeta}^{mf} + \zeta^f)}, \end{aligned} \quad (38)$$

$$\omega^f = \hat{\omega}^m + (\delta\hat{\omega}^{mf} + \delta\omega^f), \quad (39)$$

where we assumed that $\hat{\zeta}^{mf}, \zeta^f$ are sufficiently small. Thus, the uncertainty ellipsoid obtained by the flow update, $\mathcal{E}(\hat{C}^f, \hat{\omega}^f, P^f)$ is given by the center $(\hat{C}^m, \hat{\omega}^m)$ of the measurement uncertainty ellipsoid and

$$\mathcal{E}_{\mathbb{R}^6}(\hat{x}^{mf}, P^f), \quad (40)$$

where $\hat{x}^{mf} = \left[(\hat{\zeta}^{mf})^T, (\delta\hat{\omega}^{mf})^T \right]^T \in \mathbb{R}^6$.

We seek a minimal ellipsoid that contains the intersection of two uncertainty ellipsoids in \mathbb{R}^6 :

$$\mathcal{E}_{\mathbb{R}^6}(0_{6 \times 1}, P^m) \cap \mathcal{E}_{\mathbb{R}^6}(\hat{x}^{mf}, P^f) \subset \mathcal{E}_{\mathbb{R}^6}(\hat{x}, P), \quad (41)$$

where $\hat{x} = [\hat{\zeta}^T, \delta\hat{\omega}^T]^T \in \mathbb{R}^6$. We obtain \hat{x} and P as

$$\begin{aligned} \hat{x} &= L\hat{x}^{mf}, \\ P &= \beta(q)(I - L)P^m, \end{aligned}$$

where

$$\begin{aligned} \beta(q) &= 1 + q - (\hat{x}^{mf})^T (P^m)^{-1} L \hat{x}^{mf}, \\ L &= P^m (P^m + q^{-1} P^f)^{-1}. \end{aligned}$$

The constant q is chosen such that $\text{tr}[P]$ is minimized. We convert \hat{x} to points in TSO(3) using the common center $(\hat{C}^m, \hat{\omega}^m)$.

In summary, a new uncertainty ellipsoid at the $k+l$ th step is defined by

$$(C_{k+l}, \omega_{k+l}) \in \mathcal{E}(\hat{C}_{k+l}, \hat{\omega}_{k+l}, P_{k+l}), \quad (42)$$

where

$$\hat{C}_{k+l} = \hat{C}_{k+l}^m e^{S(\hat{\zeta})}, \quad (43)$$

$$\hat{\omega}_{k+l} = \hat{\omega}_{k+l}^m + \delta\hat{\omega}, \quad (44)$$

$$P_{k+l} = P. \quad (45)$$

The entire estimation procedure is repeated. The new uncertainty ellipsoid is used to predict the uncertainty ellipsoid till the next measurements are available, and the measurement update and the filtering procedures are performed. The center of the new uncertainty ellipsoid provides point estimates of the attitude and the angular velocity at the $k+l$ th step. The uncertainty matrix represents the ellipsoidal bound on uncertainty. The size of the uncertainty matrix characterizes the accuracy of the estimates. If the size is small, we conclude that the estimates are accurate. This estimation scheme is optimal since the size of the new uncertainty ellipsoid is minimized. The eigenvector of the uncertainty matrix corresponding to the maximum eigenvalue shows the direction of the maximum uncertainty.

V. NUMERICAL SIMULATION

Numerical simulation results are given for the estimation of the attitude dynamics of an uncontrolled rigid spacecraft in a circular orbit about a large central body, including gravity gradient effects. The detailed description of the on orbit spacecraft model is presented in [19].

The inertia of the spacecraft is chosen as $\overline{J} = \text{diag}[1, 2.8, 2]$, where overlines denote normalized variables. The maneuver is an arbitrary large attitude change completed in a quarter of the orbit, $\overline{T}_f = \frac{\pi}{2} \overline{s}$. The initial conditions are chosen as

$$\begin{aligned} C_0 &= \text{diag}[-1, -1, 1], \quad \overline{\omega}_0 = [2.3160, 0.4468, -0.5910]^T, \\ \hat{C}_0 &= I_{3 \times 3}, \quad \hat{\omega}_0 = [2.1160, 0.5468, -0.8910]^T. \end{aligned}$$

The corresponding initial estimation errors are $\|\zeta_0\| = 180 \text{ deg}$, $\|\delta\omega_0\| = 21.43 \frac{\pi}{180} \text{ rad/s}$. The initial uncertainty matrix is given by

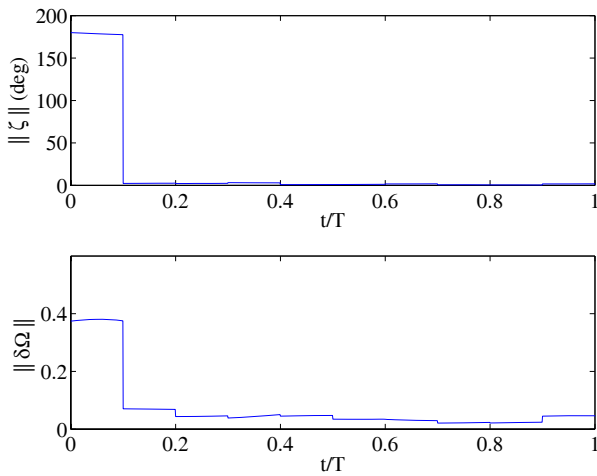
$$P_0 = 2 \text{diag} \left[\left(180 \frac{\pi}{180} \right)^2 [1, 1, 1], \left(30 \frac{\pi}{180} \right)^2 [1, 1, 1] \right],$$

so that $x_0^T P_0^{-1} x_0 = 0.7553 \leq 1$.

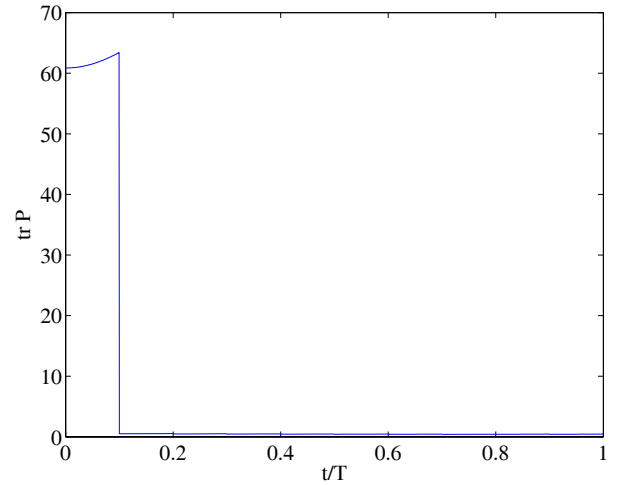
We assume that measurements are available ten times in a quarter orbit. The measurement uncertainty matrices are given by

$$S_k^i = \left(7 \frac{\pi}{180} \right)^2 I_{3 \times 3} \text{ rad}^2, \quad T_k = \left(7 \frac{\pi}{180} \right)^2 I_{3 \times 3} \text{ rad}^2 / \overline{s}^2.$$

Fig. 2 shows simulation results for a typical realization of the bounded uncertainties, where the plot on the left shows the attitude estimation error and the angular velocity estimation error, and the right plot shows the size of the uncertainty ellipsoid. The estimation errors and the size of uncertainty decrease fast after the first estimation. The terminal attitude error is less than 1 deg.



(a) Estimation error $\|\zeta_k\|$, $\|\delta\Omega_k\|$



(b) Magnitude of Uncertainty $\text{tr}[P_k]$

Fig. 2. Attitude and angular velocity estimation errors with measurements

VI. CONCLUSIONS

The attitude estimation scheme presented here has no singularities since the attitude is represented by a rotation matrix, and the structure of the group of rotation matrices is preserved since it is updated by group operations in $\text{SO}(3)$ using the Lie group variational integrator. The attitude estimator is also robust to the distribution of the uncertainty and the sensor noise, since it is based on deterministic ellipsoidal bounds on the uncertainty. The effects of process noise can be included by modifying the prediction procedure.

Although not presented in this paper, we have obtained results for the modification of this scheme to the case when angular velocity measurements are not available. We intend to extend this estimation scheme to the combined attitude control and estimation problem for a rigid body in an attitude dependent potential, with the inclusion of process noise or disturbance forces. These topics will be dealt with in a future journal paper.

VII. ACKNOWLEDGMENTS

The research of ML has been supported in part by NSF under Grant DMS-0504747. The research of NHM has been supported in part by NSF under Grant ECS-0140053.

REFERENCES

- [1] Sanyal, A. K. (2006). Optimal Attitude Estimation and Filtering Without Using Local Coordinates, Part I: Uncontrolled and Deterministic Attitude Dynamics. Submitted to the 2006 American Control Conference.
- [2] Rehinder, H., and Hu, X. (2004). Drift-free attitude estimation for accelerated rigid bodies. *Automatica*, 40(4), 653-659.
- [3] Wahba, G. (1965). A least squares estimate of satellite attitude. *SIAM Review*, 7(3), 409.
- [4] Bar-Itzhack, I. Y., and Oshman, Y. (1985). Attitude determination from vector observations: quaternion estimation. *IEEE Transactions on Aerospace and Electronic Systems*, 21(1), 128-136.
- [5] Crassidis, J. L., and Markley, F. L. (1997). A minimum model error approach for attitude estimation. *Journal of Guidance, Control and Dynamics*, 20(6), 1241-1247.
- [6] Markley, F. L. (1989). Attitude determination and parameter estimation using vector observations: theory. *Journal of the Astronautical Sciences*, 37(1), 41-58.
- [7] Shuster, M. D., and Oh, S. D. (1981). Three-axis attitude determination from vector observations. *Journal of Guidance and Control*, 4(1), 70-77.
- [8] Shuster, M. D. (1990). Kalman filtering of spacecraft attitude and the QUEST model. *Journal of the Astronautical Sciences*, 38(3), 377-393.
- [9] Barshan, B. and Durrant-Whyte, H. F. (1995). Inertial navigation systems for mobile robots. *IEEE Transactions on Robotics and Automation*, 11(3), 328-342.
- [10] Roumeliotis, S. I., Sukhatme, G. S., and Bekey, G. A. (1999). Smoother based 3D attitude estimation for mobile robot localization. In *Proceedings of 1999 IEEE International Conference on Robotics and Automation*, Vol. 3, Detroit, MI (pp. 1979-1986).
- [11] Vaganay, J., Aldon, M. J., and Fournier, A. (1993). Mobile robot attitude estimation by fusion of inertial data. In *Proceedings of the 1993 IEEE International Conference on Robotics and Automation*, Vol. 1, Atlanta, GA (pp. 277-282).
- [12] Crassidis, J. L., and Markley, F. L. (2003). Unscented filtering for spacecraft attitude estimation. *AIAA Journal of Guidance, Control, and Dynamics*, 26(4), 536-542.
- [13] Psiaki, M. L. (2000). Attitude-determination filtering via extended quaternion estimation. *AIAA Journal of Guidance, Control, and Dynamics*, 23(2), 206-214.
- [14] Milnor, J. (1963). *Morse Theory*, Princeton University Press, Princeton, NJ.
- [15] Theodor, Y., Shaked, U., and de Souza, C. E. (1994). A Game Theory Approach to Robust Discrete-Time H_∞ -Estimation. *IEEE Transactions on Signal Processing*, 42(6), 1486-1495.
- [16] Schweppe, F. C. (1968). Recursive State Estimation: Unknown but Bounded Errors and System Inputs. *IEEE Transactions on Automatic Control*, 13(1), 22-28.
- [17] Makarov, D. G., and Norton, J. P. (1996). State bounding with ellipsoidal set description of the uncertainty. *International Journal of Control*, 65(5), 847-866.
- [18] Durieu, C., Walter, E., and Polyak, B. (2001). Multi-Input Multi-Output Ellipsoidal State Bounding. *Journal of Optimization Theory and Applications*, 111(2), 273-303.
- [19] Lee, T., Leok, M., and McClamroch, N. H. (2006). Attitude Maneuvers of a Rigid Spacecraft in a Circular Orbit. Submitted to the 2006 American Control Conference. arXiv:math.OA/0509299.
- [20] Lee, T., Leok, M., and McClamroch, N. H. (2005). A Lie Group Variational Integrator for the Attitude Dynamics of a Rigid Body with Applications to the 3D Pendulum. In *Proceedings of the IEEE Conference on Control Applications*, Toronto, Canada (pp. 962-967).